Entropy decrease in Quantum Zeno Effect

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Abstract

If a measurement process is regarded as an irreversible process, then by Second law of thermodynamics the entropy should increase after any measurement process. By the same spirit a quantum system undergoing repeated measurement should show strong irreversibility leading to entropy production. On the contrary we show that in quantum Zeno effect setting the entropy of a quantum system decreases and goes to zero after a large number of measurements. We discuss the entropy change under continuous measurement model and show that entropy can decrease if we use a more accurate measuring apparatus.

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Since its inception the concept of the entropy of a state has played an important role in understanding enigmas of physics in diverse areas such as thermodynamics, quantum mechanics, information theory and recently in the area of quantum information processing. Traditionally one associates some sort of "disorder" with entropy of a physical system. The hallmark of second law of thermodynamics is that the entropy of an isolated system increases with time or remains constant. However, it increases only when a system undergoes an irreversible process (which in turn attributes an arrow of time). In terms of negentropy this should decrease for an irreversible dynamical changes in the system. Given a quantum system, if it is left undisturbed and allowed to evolve unitarily then the entropy remains constant with respect to a given preparation. This is rigidly connected with the principle of linearity of time evolution of quantum system, because one [1] could show that if the time evolution equation is non-linear then entropy of a mixture of quantum system could spontaneously decrease in a closed system.

But what about the entropy of a quantum system under observation? Since observation(measurement) on a quantum system is an irreversible process one would say that the entropy should increase as was first discussed by von Neumann [2] in formulating his measurement theory. The entropy change occuring in an isolated quantum system without measurement and entropy increase due to measurement are quite different as expounded in a lucid book by Brillouin [3]. In the later case some amount of information can be obtained as measurement involves an experiment with system and apparatus. Thus, the information

gain about a physical system by measuring a complete set of commuting observables must be paid for in negative entropy(negentropy). This would mean that if we do repeated measurements on a quantum system we would gain more and more information and the quantum system should show a strong irreversibility leading to increase of entropy. In other words the above reasoning would inevitably lead us to say that frequent observation on a quantum system system should increase the entropy of the system.

In this paper we investigate the entropy of a quantum system under repeated observation. To be specific we look for the entropy change when the unitary evolution of a quantum system is interrupted by sequence of measurements (we call such a dynamics quantum Zeno dynamics (QZD)). This would also answer the question:how does entropy change when we tend to know more and more about the evolution of a quantum system. Contary to aforesaid paragraph, we find that the repeated measurements on a quantum system tends to decrease the entropy of a quantum system and under continuous observation the entropy goes to zero. This can be proved within von Neumann's collapse mechanism and unitary time evolution. Also, the same can be proved using a continuous measurement model where the effective evolution is non-unitary. In the sequel, we touch questions like does the entropy depend upon the precision with which an apparatus measures the observable of the system under study. With better measuring apparatus can we be able to get much detailed information about the system? Does this decrease the entropy? Since there is considerable interest in the theoretical and experimental issues of quantum Zeno effect(QZE) we believe that the present results will be of importance in answering certain subtle issues like entropy and information under repeated observation.

The quantum Zeno effect (QZE) was originally discovered for an unstable quantum system by Misra and Sudarshan [4]. For a coherent system QZE says that if we prepare the system initially in an eigenstate of some observable and repeatedly disturbes the unitary evolution of the system by successive measurements, then the quantum transition to other states can be completely suppressed. Recently, Itano et al [5] have carried out an experiment to test the QZE following a proposal of Cook [6]. This experiment gave rise to debates over the fundamental issues of quantum theory which have been discussed by several authors [7–9] and also by the present author [10]. In a continuing debate the present author and Lawande [11] has questioned the necessity of Schrödinger time evolution and argued that QZE could be observed in non-linear quantum systems. Let us consider a quantum system which has been prepared in the eigenstate of some observable A that we are interested in measuring. The observable A has a discrete spectrum $\{a_n\}$ and a complete set of eigenstates $\{|\psi_n\rangle\}$. In the absence of any measurement the system at a later time t will make transition to other states under the action of some unitary operator and the probabilities are distributed according to $p_n = |c_n|^2$. Thus the state at time $t = 0$ evolves to a state at time t given by

$$|\psi(t)\rangle = U(t)|\psi(0)\rangle = \sum_n c_n(t)|\psi_n\rangle \tag{1}$$

One can associate an entropy of the quantum system given these probability distributions$\{p_1, p_2, ...p_n\}$ as given by

$$S(t) = -\sum_n p_n(t)\log p_n(t) \tag{2}$$

which is called Shannon entropy in information theory. This entropy depends on the initial preparation stage of a quantum system. We will discuss the effect of repeated measurements

on the Shannon entropy with respect to a given preparation stage. Since we have prepared our system in the eigenstate of some observable A the amplitudes can be written as

$$c_n(t) = \langle\psi_n|e^{-iHt/\hbar}|\psi_n\rangle = \sum_m e^{-iE_m t/\hbar} P_{nm} \tag{3}$$

where $P_{nm} = |\langle\psi_n|\phi_m\rangle|^2$ and $\{|\phi_m\rangle\}$ being the basis in which H diagonalises. The matrix P_{nm} are transition probability matrix elements some times called "doubly stochastic matrix" which satsify $\sum_n P_{nm} = I = \sum_m P_{nm}$. Now the probability distributions are given by

$$p_n(t) = \sum_{mk} e^{-i(E_m-E_k)t/\hbar} P_{nm}P_{nk}. \tag{4}$$

Therefore, the Shannon entropy corresponding to the preparation of the system in the eigenstate of A leads to the expression

$$S(t) = -\sum_n \Big[\sum_{mk} \cos\omega_{mk}tP_{nm}P_{nk}\log(\sum_{mk}\cos\omega_{mk}tP_{nm}P_{nk})\Big] \tag{5}$$

where $\omega_{mk} = (E_m - E_k)t/\hbar$ is the transition frequency between two energy levels and we have dropped the imaginary part of (3) because $p_n(t)$ are real quantities. On the other hand if we could prepare the system in the eigenstate of the Hamiltonian then the initial and final entropy remain the same. But this preparation stage is not interesting from the standpoint of QZE becuase the probability of finding the system in the nth eigenstate is always unity irrespective of repeated measurements. Therefore, it is essential that we should prepare our system in eigenstate of some observable which does not commute with the Hamiltonian of the system [10].

We investigate the Shannon entropy of the system when the unitary evolution during the time interval $[0, T]$ is interrupted by von Neumann measurements such that one performs a series of measurements at times $\tau, 2\tau \ldots (N-1)\tau, N\tau = T$. During the short time interval $[0, \tau]$ the system evolves unitarily. The sequence of measurements that are carried out are idealised to be discrete and instantaneous.

After performing a von Neumann measurement at time τ the probability of finding the system in the nth state is given by

$$p_n(\tau) = 1 - \frac{\tau^2}{2}\sum_{mk}{\omega_{mk}}^2 P_{nm}P_{nk} \tag{6}$$

When the system undergoes repeated measurements N number of times the probability of finding the system in the nth state is given by

$$p_n(T) = [p_n(\tau)^N] = (1 - \frac{T^2}{2N}\sum_{mk}{\omega_{mk}}^2 P_{nm}P_{nk})^N \tag{7}$$

Thus the probabilities are distributed according to above rule after N number of measurements and hence the Shannon entropy of the system is given by

$$S_n(T) = \frac{T^2}{2N}\sum_n\Big[\sum_{mk}{\omega_{mk}}^2 P_{nm}P_{nk}exp(-\frac{T^2}{2N}\sum_{mk}{\omega_{mk}}^2 P_{nm}P_{nk})\Big] \tag{8}$$

showing a clear dependence on the number of measurements performed on the quantum system. In the above expression we have used the large N limit of the probability distributions. From (5) one can conclude that as the number of measrements tend to infinity the Shannon entropy of the system goes to zero. This is contrary to our intution that large number of meaurements should be a signature of strong irreversibility leading to entropy increase. Rather, we find that the entropy of the system decreases (as it starts from a finite non-zero value and goes to zero) when we tend to know more and more about the evolution of a quantum system.

The same result can also be proved within a continuous measurement model [12,13] which has been often invoked to simulate the quantum Zeno effect without using von Neumann's projection postulate. These models are different from the continuous measurement models described by stochastic equation for the quantum sysytem and quantum trajectory approach. The measurement process in the above case is described by a non-unitary evolution equation. This is done usually by taking an effective Hamiltonian which is non-Hermitian in nature. This kind of model has been very useful in proving new results and predicting new quantum effect such as quantum Zeno Phase effect (QZPE) [15,16]. For details one can refer to [12,15]. We briefly recall that the evolution equation for a quantum system undergoing continuous measurement of some observable A is given by

$$i\hbar \frac{d|\psi(t)\rangle}{dt} = \Big[H - i\hbar f g(\frac{(A-a)}{\Delta a})\Big]|\psi(t) > \tag{9}$$

where H is the free Hamiltonian, a is the result of the measurement that the apparatus reads, Δa is the the accuracy of the measurement process and f is the rate of information gain on the observable of the system. The function $g(\frac{(A-a)}{\Delta a})$ takes care the interaction between the system and apparatus in a parameter dependent way. The physical basis for such a phenomenological equation has been given on restricted path integral approach [14] and also heuristically in [13].

Let us prepare our system as before in the eigenstate of the observable A and follow the continuous evolution equation, then the time evolution can be given by

$$|\psi(t)\rangle = \exp[-i(H - i\hbar f g(\frac{(A-a)}{\Delta a}))t/\hbar]|\psi_n\rangle. \tag{10}$$

Now the probability amplitude in the nth state would be given by

$$c_n(t) = \langle\psi_n|e^{-i(H-i\hbar f u(A;a,\Delta a))t/\hbar}|\psi_n\rangle. \tag{11}$$

where $u(A;a,\Delta a) = g(\frac{(A-a)}{\Delta a})$. The above equation is in general difficult to simplify because the observable A does not commute with the Hamiltonian. But we can use Campbell-Beker-Hausdorff formula to simplify it to some extent. Without loss of generality for our purpose we assume that the commutator of H and A commutes with H and A. In that case we can express the amplitudes as

$$c_n(t) = e^{-fg(\frac{(a_n-a)}{\Delta a})t}\langle\psi_n|e^{-iHt}e^{-i/2ft[H,u(A)]}|\psi_n\rangle. \tag{12}$$

Therefore, the probabilities are now distributed according to

$$p_n(t) = e^{-2fg(\frac{(a_n-a)}{\Delta a})t} |V_{nn}|^2. \tag{13}$$

where $V_{nn} = \langle \psi_n | e^{-iHt} e^{-i/2ft[H,u(A)]} | \psi_n \rangle$ is the matrix element involving all other operators. With this distribution we can see that Shannon entropy is given by

$$S(t) = 2f \sum_n u_n e^{-2fu_n} {V_{nn}}^2 - \sum_n e^{-2fu_n} {V_{nn}}^2 \log {V_{nn}}^2. \tag{14}$$

From the above expression one can clearly see that when we obtain more information about the observable, i.e., in the limit of high frequency of measurement the Shannon entropy goes to zero. It is interesting to note that if we consider an observable which commutes with Hamiltonian then the term ${V_{nn}}^2$ is unity and the entropy takes a simple form $S = 2f \sum_n u_n e^{-2fu_n}$, which also goes to zero in the limit of high frequency of measurements. Note that in the continuous measurement model we can talk of a commuting observable with Hamiltonian and can still have an interesting physical situation.

This model also provides answer to the question: Does the entropy depend on the accuracy of the measuring apparatus? Does it decrease with increasing the accuracy of the device? The answer is yes. As we can see when the accuracy of the device increases the term $g(A; a, \Delta a)$ tends to infinite and then the entropy again goes to zero. This is possible because the function $g(A; a, \Delta a)$ is a positive function of its argument, i.e., $g(x) \geq 0, g(0) = 0$. Generally, it is assumed that $g(x) = x^2$ which gives a gussian type function in the time evolution operator.

To conclude this paper we have shown that the Shannon entropy of a quantum system decreases, and, in fact, goes to zero when it is interrupted by a large sequence of measurements of the von Neumann type. The same result is proved within a continuous measurement model. This is somewhat counter intutive that measurement which is supposed to increase the entropy of the system, the repeated measurements do the opposite. This result raises several questions which are related to quantum measurement theory, second law of thermodynamics and the nature of entropy itself. Is it that a quantum system which has undergone several measurements in the past occupies a lowest favourable state (since entropy is minimum). Is it that all the well organised things that we see arround are results of continuous measurements that our whole universe is undergoing? We hope that the new effect of entropy decrease in the quantum zeno dynamics setting may be another way to achive lowest entropy states.

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